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Weakly E-unitary locally inverse semigroups

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Abstract

We prove that each weakly E-unitary locally inverse semigroup is embeddable in a restricted semidirect product of a normal band by a completely simple semigroup and, equivalently, in a Pastijn product of a normal band by a completely simple semigroup.

Keywords: Locally inverse semigroup; Weak E-unitariness; Restricted semidirect product; Pastijn product

1. Introduction

The structure of locally inverse semigroups—formerly called also pseudo-inverse semigroups—has been intensively studied, mainly in the 80s. The most significant approaches and results are due to D.B. McAlister, K.S.S. Nambooripad and F. Pastijn. For a bibliography, the reader is referred to [3] and [9]. Our results relate mostly to those of F. Pastijn.

In [8], F. Pastijn proved a structure theorem for perfect—called also elementary—rectangular bands of E-unitary inverse semigroups which generalizes D.B. McAlister's P-theorem. The construction he applies is reminiscent of a semidirect product of a semilattice by a completely simple semigroup. Perfect rectangular bands of E-unitary

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inverse semigroups constitute an important subclass of weakly E-unitary locally inverse semigroups. The latter class was introduced and studied by R. Veeramony [12]. He established a generalization of the P-theorem for weakly E-unitary locally inverse semigroups with injective structure mappings by making use of inductive groupoids acting on partially ordered sets on both sides. The notion of a weakly E-unitary locally inverse semigroup was—independently—defined also by J. Kad'ourek [5]. The property that their least completely simple semigroup congruence is idempotent pure, together with the results mentioned and the embedding theorem of the second author in [11] for E-unitary generalized inverse semigroups, might remind the reader of the question whether each weakly E-unitary locally inverse semigroup is embeddable in a semidirect-like product of a normal band by a completely simple semigroup.

The main result of the paper (Section 4) answers this question in the positive. Notice that the weakly E-unitary locally inverse semigroups with injective structure mappings are just the weakly E-unitary straight locally inverse semigroups, cf. Proposition 4.9.

In Section 3 it is proved that two semidirect-like constructions playing an important role in the theory of regular semigroups—the restricted semidirect product by a completely simple semigroup [1] and the Pastijn product ([5], cf. also [8])—are equivalent from the point of view of which regular semigroups are embeddable into them. Moreover, it is noticed that the Pastijn product is a self-dual construction.

The main result of Section 4 was proved by the first author in the special case of weakly E-unitary straight locally inverse semigroups by making use of another setting. The simpler construction providing a more general embedding theorem presented in the paper is due to the second author.

2. Preliminaries

For the standard notions and notation in semigroup theory the reader is referred to [3].

Given a set X, we denote the free semigroup and the free monoid on X by X^+ and X^*, respectively. If $w \in X^+$ then $C(w)$ will stand for the *content* of w, that is, for the set of the elements of X occurring in w. Furthermore, $h(w)$ is used to denote the *head* of w, that is, the element of X appearing in w first (when reading it from the left).

We introduce the following notation for the four varieties of normal bands containing semilattices:

S – semilattices,
LN – left normal bands,
RN – right normal bands,
N – normal bands.

Let **V** be a variety of normal bands containing semilattices. Define

$$\mathbf{V}_0 = \begin{cases} \mathbf{LN} & \text{if } \mathbf{LN} \subseteq \mathbf{V}, \\ \mathbf{S} & \text{otherwise}, \end{cases}$$

and define $\mathbf{V}_1$ dually. It is well known that $\mathbf{V} = \mathbf{V}_0 \vee \mathbf{V}_1$.

Let S be a semigroup. If $I \times \Lambda$ is a rectangular band and S is the disjoint union of its subsemigroups $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$, such that $S_{i\lambda} S_{j\mu} \subseteq S_{i\mu}$ for all $(i, \lambda), (j, \mu) \in I \times \Lambda$ then we say that S is a *rectangular band* $I \times \Lambda$ *of its subsemigroups* $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$. If, moreover, $S_{i\lambda} S_{j\mu} = S_{i\mu}$ for any $(i, \lambda), (j, \mu) \in I \times \Lambda$ then we term S a *perfect rectangular band* $I \times \Lambda$ *of its subsemigroups* $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$.

If S is a rectangular band $I \times \Lambda$ of its subsemigroups $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$, then the mapping $\pi : S \to I \times \Lambda,\ s \mapsto (i, \lambda)$ if $s \in S_{i\lambda}$ is clearly a homomorphism. For brevity, we shall denote $E_{S_{i\lambda}}$ by $E_{i\lambda}, (i, \lambda) \in I \times \Lambda$.

Let ρ be a congruence relation on a regular semigroup S. If $e\,\rho\,a$ implies $a \in E$ for every $e \in E$ and $a \in S$, then ρ is termed *idempotent pure*. If S/ρ is a completely simple semigroup [rectangular band] then ρ is called a *completely simple semigroup congruence* [*rectangular band congruence*]. On any regular semigroup S, there exists a least completely simple semigroup congruence [rectangular band congruence]. We denote this congruence by ξ_S $[\beta_S]$ or briefly by ξ $[\beta]$. Obviously, we have $\xi \subseteq \beta$ for any regular semigroup S. In particular, if S is a rectangular band of its subsemigroups then the congruence $\ker \pi$ on S induced by π is contained in β.

Given a completely simple semigroup T, we shall find it convenient to choose and fix a Rees matrix representation $\mathcal{M}[G; I, \Lambda; P]$ for T where P is a normal sandwich matrix, that is, there exists $0 \in I \cap \Lambda$ such that we have $p_{\lambda 0} = p_{0i} = 1$, the identity of G, for every $i \in I$ and $\lambda \in \Lambda$. For notational convenience we identify T with $\mathcal{M}[G; I, \Lambda; P]$, that is, we suppose that $T = \mathcal{M}[G; I, \Lambda; P]$.

If S is a regular semigroup and $S/\xi = \mathcal{M}[G; I, \Lambda; P]$ then this representation induces the following decomposition of S. For every $(i, \lambda) \in I \times \Lambda$, put

$$S_{i\lambda} = \bigl\{ s \in S\colon\ s\xi = (i, g, \lambda) \text{ for some } g \in G \bigr\}.$$

Obviously, S is a rectangular band $I \times \Lambda$ of its subsemigroups $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$.

A locally inverse semigroup is called by F. Pastijn and M. Petrich [10] *straight* if its maximal subsemilattices are pairwise disjoint. An important subclass of straight locally inverse semigroups is that of perfect rectangular bands of inverse semigroups. Let S be a straight locally inverse semigroup. It is well known from the results by F. Pastijn [8] and [9] that S is a rectangular band $I \times \Lambda$ of its subsemigroups $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$, where $E_{i\lambda}, (i, \lambda) \in I \times \Lambda$, are the maximal subsemilattices of S. Furthermore, $\ker \pi = \beta$, $E_i = \bigcup_{\lambda \in \Lambda} E_{i\lambda}$ is a right normal band and $E_\lambda = \bigcup_{i \in I} E_{i\lambda}$ is a left normal band for every $i \in I$ and for every $\lambda \in \Lambda$, respectively.

The latter property allows us to introduce the following generalizations of the notion of a straight locally inverse semigroup. We term a locally inverse semigroup S *right straight* [*left straight*] if it is a rectangular band $I \times \Lambda$ of its subsemigroups $S_{i\lambda}, (i, \lambda) \in I \times \Lambda$, where $E_i = \bigcup_{\lambda \in \Lambda} E_{i\lambda}$ is a right normal band for every $i \in I$ $[E_\lambda = \bigcup_{i \in I} E_{i\lambda}$ is a left normal band for every $\lambda \in \Lambda]$. It is straightforward to see that a locally inverse semigroup is straight if and only if it is left and right straight. Notice also that a regular subsemigroup of a straight [right straight, left straight] locally inverse semigroup is of the same property.

The notion of a weakly E-unitary locally inverse semigroup was introduced by R. Veeramony [12] and by J. Kad'ourek [5] as follows. A locally inverse semigroup S

is called *weakly E-unitary* if E is a filter in S with respect to the natural partial order, that is, if $e \leqslant a$ implies $a \in E$ for every $e \in E$ and $a \in S$. It is clear from the definition that a regular subsemigroup in a weakly E-unitary locally inverse semigroup is weakly E-unitary.

The following characterization of weakly E-unitary locally inverse semigroups is proved in [5].

Result 2.1. *A locally inverse semigroup S is weakly E-unitary if and only if the congruence ξ on S is idempotent pure.*

In particular, if S is a weakly E-unitary straight locally inverse semigroup then the idempotent ξ-classes are just the maximal subsemilattices of S.

Now we recall two constructions which produce locally inverse semigroups from normal bands and completely simple semigroups. Actually, we define these constructions in a more general setting.

Let K be any semigroup and let T be a completely simple semigroup. Let us choose a Rees matrix representation for T, say $T = \mathcal{M}[G; I, \Lambda; P]$. Suppose that G acts on K by automorphisms on the left—briefly, G acts on K—, that is, for each $g \in G$, an automorphism $a \mapsto {}^{g}a$ of K is given such that ${}^{h}({}^{g}a) = {}^{hg}a$ for every $a \in K$ and $g, h \in G$. Define a multiplication on $K \times T$ by

$$\bigl(a, (i, g, \lambda)\bigr)\bigl(b, (j, h, \mu)\bigr) = \bigl(a \cdot {}^{gp_{\lambda j}}b, (i, gp_{\lambda j}h, \mu)\bigr).$$

It is straightforward to check that in this way a semigroup is obtained. We call it a *Pastijn product of K by T* and denote it by $K \odot T$. This construction was introduced in [8] and was applied also in [5] and [6]. It was noticed in [5] that $K \odot T$ does not essentially depend on the Rees matrix representation of T chosen.

We shall need the following properties of this construction.

Result 2.2. *Let K be a normal band and $T = \mathcal{M}[G; I, \Lambda; P]$ a completely simple semigroup. Assume that G acts on K.*

(i) *$K \odot T$ is a perfect rectangular band $I \times \Lambda$ of the E-unitary generalized inverse subsemigroups*

$$(K \odot T)_{i\lambda} = \bigl\{\bigl(a, (i, g, \lambda)\bigr) \in K \odot T\colon\ a \in K \text{ and } g \in G\bigr\}, \quad (i, \lambda) \in I \times \Lambda.$$

Consequently, $K \odot T$ is a weakly E-unitary locally inverse semigroup.

(ii) *If K is a left [right] normal band then $K \odot T$ is a left [right] straight locally inverse semigroup, and $(K \odot T)_{i\lambda}$ is an E-unitary $\mathcal{R}$-unipotent [$\mathcal{L}$-unipotent] generalized inverse semigroup for every $(i, \lambda) \in I \times \Lambda$.*

(iii) *If K is a semilattice then $K \odot T$ is a straight locally inverse semigroup and $(K \odot T)_{i\lambda}$ is an E-unitary inverse semigroup for every $(i, \lambda) \in I \times \Lambda$.*

(iv) *The second projection is a surjective homomorphism of $K \odot T$ to T which induces an idempotent pure congruence on $K \odot T$.*

The other construction we consider is a specialization of the restricted semidirect product introduced by K. Auinger and L. Polák [1].

Let K be any semigroup and let T be a completely simple semigroup. Assume that T acts on K by endomorphisms on the left—briefly, T acts on K—, that is, for each $t \in T$, an endomorphism $a \mapsto {}^t a$ of K is given such that ${}^u({}^t a) = {}^{ut}a$ for every $a \in K$ and $t, u \in T$. Define a multiplication on the set

$$\{(a,t) \in K \times T\colon {}^e a = a \text{ for some } e \in E_T \text{ with } e\,\mathcal{R}\,t\}$$

by

$$(a,t)(b,u) = \left(a \cdot {}^t b, tu\right).$$

It is easy to check that, since T is completely simple, this product coincides with that in [1], and so a semigroup is obtained which we call a *restricted semidirect product of K by T* and denote $K *_{rr} T$. Notice that if ${}^e a = a$ for some $e \in E_T$ with $e\,\mathcal{R}\,t$ then ${}^{e'}a = {}^{e'}({}^e a) = {}^{e'e}a = {}^e a = a$ for any $e' \in E_T$ with $e'\,\mathcal{R}\,t$.

Let us note that if K is a regular semigroup then $K *_{rr} T$ is also regular, and so $K *_{rr} T$ is a regular subsemigroup in the so-called regular semidirect product $K *_r T$ of K by T corresponding to the same action of T on K. The latter product was introduced by P.R. Jones and P.G. Trotter [4] and was also investigated by the authors [2].

The analogue of Result 2.2 holds for a restricted semidirect product of a normal band by a completely simple semigroup. We do not formulate it because it will follow from the first result in the next section stating that each restricted semidirect product of a semigroup K by a completely simple semigroup T is isomorphic to a Pastijn product of a subsemigroup of K by T.

3. The Pastijn product and the restricted semidirect product

The aim of this section is to prove that, from the point of view of embeddability, the two constructions, the Pastijn product and the restricted semidirect product by a completely simple semigroup, are equivalent. Furthermore, we show that the Pastijn product is a self-dual construction.

First we establish that each restricted semidirect product by a completely simple semigroup is isomorphic to a Pastijn product.

Theorem 3.1. *Let K be a* (*regular*) *semigroup and T a completely simple semigroup. Each restricted semidirect product of K by T is isomorphic to a Pastijn product of a* (*regular*) *subsemigroup of K by T.*

Proof. Let $T = \mathcal{M}[G; I, \Lambda; P]$ where P is a normal sandwich matrix. Assume that T acts on K, and so a restricted semidirect product $K *_{rr} T$ is defined. Put $G_0 = \{(0, g, 0) \in T\colon\ g \in G\}$ and $K_0 = {}^{(0,1,0)}K$. Clearly, K_0 is a subsemigroup in K, and it is regular if K is regular. Obviously, G_0 is a subgroup in T and it acts on K by endomorphisms.

Moreover, we have ${}^{t}a \in K_0$ for every $t \in G_0$ and $a \in K_0$. Therefore this action induces an action of G_0 on K_0 by endomorphisms. What is more, this is an action by automorphisms since the identity element $(0,1,0)$ of G_0 acts on K_0 identically. Since the mapping $G \to G_0,\ g \mapsto (0,g,0)$ is an isomorphism, we see that the equality ${}^{g}a = {}^{(0,g,0)}a$ defines an action of G on K_0. This action determines a Pastijn product $K_0 \odot T$.

Define a mapping $\phi : K *_{rr} T \to K_0 \odot T$ by putting

$$\big(a, (i,g,\lambda)\big)\phi = \big({}^{(0,1,0)}a, (i,g,\lambda)\big).$$

We verify that ϕ is an isomorphism. Let $(a,(i,g,\lambda)), (b,(j,h,\mu)) \in K *_{rr} T$. To prove injectivity of ϕ, suppose that $(a,(i,g,\lambda))\phi = (b,(j,h,\mu))\phi$. Then $t = (i,g,\lambda) = (j,h,\mu)$, in particular, $i = j$, and ${}^{(0,1,0)}a = {}^{(0,1,0)}b$. Since, by the definition of $K *_{rr} T$, we have $a = {}^{(i,1,0)}a$ and $b = {}^{(i,1,0)}b$, we obtain that

$$a = {}^{(i,1,0)}a = {}^{(i,1,0)(0,1,0)}a = {}^{(i,1,0)}\big({}^{(0,1,0)}a\big) = {}^{(i,1,0)}\big({}^{(0,1,0)}b\big) = {}^{(i,1,0)(0,1,0)}b = {}^{(i,1,0)}b = b.$$

Thus ϕ is, indeed, injective.

Now we show that it is a homomorphism. We have

$$\begin{aligned}\big(\big(a,(i,g,\lambda)\big)\big(b,(j,h,\mu)\big)\big)\phi &= \big(a \cdot {}^{(i,g,\lambda)}b, (i, gp_{\lambda j}h, \mu)\big)\phi \\ &= \big({}^{(0,1,0)}\big(a \cdot {}^{(i,g,\lambda)}b\big), (i, gp_{\lambda j}h, \mu)\big) \\ &= \big({}^{(0,1,0)}a \cdot {}^{(0,1,0)(i,g,\lambda)}b, (i, gp_{\lambda j}h, \mu)\big).\end{aligned}$$

Since $(b,(j,h,\mu)) \in K *_{rr} T$, we have

$$b = {}^{(j,1,0)}b = {}^{(j,1,0)}\big({}^{(0,1,0)}b\big).$$

Therefore

$${}^{(0,1,0)(i,g,\lambda)}b = {}^{(0,g,\lambda)}b = {}^{(0,g,\lambda)}\big({}^{(j,1,0)}\big({}^{(0,1,0)}b\big)\big) = {}^{(0,gp_{\lambda j},0)}\big({}^{(0,1,0)}b\big) = {}^{gp_{\lambda j}}\big({}^{(0,1,0)}b\big)$$

in K and in K_0, respectively. Thus we see that

$$\begin{aligned}\big(\big(a,(i,g,\lambda)\big)\big(b,(j,h,\mu)\big)\big)\phi &= \big({}^{(0,1,0)}a \cdot {}^{gp_{\lambda j}}\big({}^{(0,1,0)}b\big), (i, gp_{\lambda j}h, \mu)\big) \\ &= \big({}^{(0,1,0)}a, (i,g,\lambda)\big)\big({}^{(0,1,0)}b, (j,h,\mu)\big) \\ &= \big(a,(i,g,\lambda)\big)\phi \cdot \big(b,(j,h,\mu)\big)\phi.\end{aligned}$$

Finally, we verify that ϕ is surjective. Let $(c,(i,g,\lambda))$ be any element in $K_0 \odot T$. Define $a = {}^{(i,1,0)}c$. Then ${}^{(i,1,0)}a = {}^{(i,1,0)^2}c = {}^{(i,1,0)}c = a$, and so $(a,(i,g,\lambda)) \in K *_{rr} T$. Furthermore, ${}^{(0,1,0)}a = {}^{(0,1,0)(i,1,0)}c = {}^{(0,1,0)}c = c$. Hence $(c,(i,g,\lambda)) = (a,(i,g,\lambda))\phi$, completing the proof. $\square$

Now we prove that each Pastijn product is embeddable in a restricted semidirect product by a completely simple semigroup.

Theorem 3.2. *Let K be an arbitrary semigroup and T a completely simple semigroup. Each Pastijn product of K by T is embeddable in a restricted semidirect product of a direct power of K by T.*

Proof. Let $T = \mathcal{M}[G; I, \Lambda; P]$ where P is a normal sandwich matrix. Assume that the group G acts on K, and denote by $K \odot T$ the Pastijn product determined. We use a wreath product type construction to define the restricted semidirect product needed in the proof.

Let R_0 be the $\mathcal{R}$-class of $(0, 1, 0)$ in T, and consider the direct power K^{R_0} of K. For any $t \in T$ and $f \in K^{R_0}$, let ${}^t f \in K^{R_0}$ be defined by $x({}^t f) = (xt)f$, $x \in R_0$. It is straightforward to see that this defines an action of T on K^{R_0}, and so a restricted semidirect product $K^{R_0} *_{rr} T$ is defined. For any $(i, a) \in I \times K$, consider the mapping $f_{i,a} \in K^{R_0}$ given by $(0, k, \kappa) f_{i,a} = {}^{kp_{\kappa i}}a$. Notice that ${}^{(i,1,0)}f_{i,a} = f_{i,a}$ since, for any $(0, k, \kappa) \in R_0$, we have

$$\begin{aligned}(0, k, \kappa)\big({}^{(i,1,0)}f_{i,a}\big) &= \big((0, k, \kappa)(i, 1, 0)\big)f_{i,a} = (0, kp_{\kappa i}, 0)f_{i,a} \\ &= {}^{kp_{\kappa i}p_{0i}}a = {}^{kp_{\kappa i}}a = (0, k, \kappa)f_{i,a}.\end{aligned}$$

Hence we see that $(f_{i,a}, (i, g, \lambda)) \in K^{R_0} *_{rr} T$ for any $a \in K$ and $(i, g, \lambda)) \in T$.

Now we verify that the mapping $\psi : K \odot T \to K^{R_0} *_{rr} T$ defined by

$$\big(a, (i, g, \lambda)\big)\psi = \big(f_{i,a}, (i, g, \lambda)\big)$$

is an embedding. Let $(a, (i, g, \lambda)), (b, (j, h, \mu)) \in K \odot T$. First we show that ψ is injective. If $(a, (i, g, \lambda))\psi = (b, (j, h, \mu))\psi$ then $(i, g, \lambda) = (j, h, \mu)$, in particular, $i = j$, and $f_{i,a} = f_{i,b}$. Then we have ${}^{kp_{\kappa i}}a = {}^{kp_{\kappa i}}b$ for every $k \in G$ and $\kappa \in \Lambda$. In particular, $a = {}^{1p_{0i}}a = {}^{1p_{0i}}b = b$, which shows that ψ is injective.

Now we verify that ψ is a homomorphism. By definition, we have

$$\begin{aligned}\big(\big(a, (i, g, \lambda)\big)\big(b, (j, h, \mu)\big)\big)\psi &= \big(a \cdot {}^{gp_{\lambda j}}b, (i, gp_{\lambda j}h, \mu)\big)\psi \\ &= \big(f_{i,a \cdot {}^{gp_{\lambda j}}b}, (i, gp_{\lambda j}h, \mu)\big)\end{aligned}$$

and

$$\begin{aligned}\big(a, (i, g, \lambda)\big)\psi \cdot \big(b, (j, h, \mu)\big)\psi &= \big(f_{i,a}, (i, g, \lambda)\big)\big(f_{j,b}, (j, h, \mu)\big) \\ &= \big(f_{i,a} \cdot {}^{(i,g,\lambda)}f_{j,b}, (i, gp_{\lambda j}h, \mu)\big).\end{aligned}$$

Since

$$\begin{aligned}(0, k, \kappa)\big(f_{i,a} \cdot {}^{(i,g,\lambda)}f_{j,b}\big) &= (0, k, \kappa)f_{i,a} \cdot (0, kp_{\kappa i}g, \lambda)f_{j,b} = {}^{kp_{\kappa i}}a \cdot {}^{kp_{\kappa i}gp_{\lambda j}}b \\ &= {}^{kp_{\kappa i}}\big(a \cdot {}^{gp_{\lambda j}}b\big) = (0, k, \kappa)f_{i,a \cdot {}^{gp_{\lambda j}}b}\end{aligned}$$

for every $(0, k, \kappa) \in R_0$, we see that $f_{i,a} \cdot {}^{(i,g,\lambda)}f_{j,b} = f_{i,a \cdot {}^{gp_{\lambda j}}b}$, and therefore ψ is a homomorphism. The proof is complete. □

In the rest of this section we establish that the Pastijn product is a self-dual construction, that is, a left-right dual of a Pastijn product of a semigroup K by a completely simple semigroup T is isomorphic to a Pastijn product of K by T.

First, let us define the left-right dual of a Pastijn product.

Let K be any semigroup and let T be a completely simple semigroup. Let us choose a Rees matrix representation for T, say $T = \mathcal{M}[G; I, \Lambda; P]$. Suppose that G acts on K by automorphisms on the right, that is, for each $g \in G$, an automorphism $a \mapsto a^g$ of K is given such that $(a^g)^h = a^{gh}$ for every $a \in K$ and $g, h \in G$. Define a multiplication on $T \times K$ by

$$\big((i, g, \lambda), a\big)\big((j, h, \mu), b\big) = \big((i, gp_{\lambda j}h, \mu), a^{p_{\lambda j}h} \cdot b\big).$$

The semigroup obtained is called a *dual Pastijn product of K by T* and is denoted by $T \odot_d K$.

Theorem 3.3. *Let K be a semigroup and let $T = \mathcal{M}[G; I, \Lambda; P]$ be a completely simple semigroup. Suppose that the group G acts on K by automorphisms on the right. Define a left action of G on K such that ${}^g a = a^{g^{-1}}$ for every $g \in G$ and $a \in K$. Then the mapping $\delta : T \odot_d K \to K \odot T$ defined by $((i, g, \lambda), a)\delta = (a^{g^{-1}}, (i, g, \lambda))$ is an isomorphism of the dual Pastijn product of K by T corresponding to the given right action onto the Pastijn product of K by T corresponding to the left action defined.*

Proof. It is clear that the mapping δ is bijective. Now we verify that it is a homomorphism. Let $((i, g, \lambda), a), ((j, h, \mu), b) \in T \odot_d K$. Then we have

$$\begin{aligned}
\big(\big((i, g, \lambda), a\big)\big((j, h, \mu), b\big)\big)\delta &= \big((i, gp_{\lambda j}h, \mu), a^{p_{\lambda j}h} \cdot b\big)\delta \\
&= \big(\big(a^{p_{\lambda j}h} \cdot b\big)^{(gp_{\lambda j}h)^{-1}}, (i, gp_{\lambda j}h, \mu)\big) \\
&= \big(a^{g^{-1}} \cdot b^{(gp_{\lambda j}h)^{-1}}, (i, gp_{\lambda j}h, \mu)\big) \\
&= \big(a^{g^{-1}} \cdot \big(b^{h^{-1}}\big)^{(gp_{\lambda j})^{-1}}, (i, gp_{\lambda j}h, \mu)\big) \\
&= \big(a^{g^{-1}} \cdot {}^{gp_{\lambda j}}\big(b^{h^{-1}}\big), (i, gp_{\lambda j}h, \mu)\big) \\
&= \big(a^{g^{-1}}, (i, g, \lambda)\big)\big(b^{h^{-1}}, (j, h, \mu)\big) \\
&= \big((i, g, \lambda), a\big)\delta \cdot \big((j, h, \mu), b\big)\delta. \quad \square
\end{aligned}$$

4. Embedding of weakly E-unitary locally inverse semigroups

This section is devoted to proving our main result.

Theorem 4.1. *Each weakly E-unitary locally inverse semigroup S is embeddable in a restricted semidirect product* [*Pastijn product*] *of a normal band B by a completely simple* *semigroup. In particular, if S is straight then B can be chosen to be a semilattice, and if S is left* [*right*] *straight then B can be chosen to be a left* [*right*] *normal band.*

First we introduce a partial semigroup which will play a crucial role in the embedding.

Let S be a weakly E-unitary locally inverse semigroup. By Result 2.1, ξ is idempotent pure on S. Consider S as a rectangular band $I \times \Lambda$ of its subsemigroups $S_{i\lambda}$, $(i, \lambda) \in I \times \Lambda$, where π has β as its kernel (cf. Section 2). Put $T = S/\xi$.

Define

$$\mathcal{X} = \{(a, t) \in S \times T\colon\ a\xi \,\mathcal{L}\, t\}.$$

If $x = (a, t) \in \mathcal{X}$ then we denote a by $\ell(x)$ and t by $\omega(x)$. We call $\ell(x)$ the *label* of x.

Define a partial multiplication $\circ$ on $\mathcal{X}$ as follows. For any $(a, t), (b, u) \in \mathcal{X}$, the product

$$(a, t) \circ (b, u) \text{ is defined} \quad \text{if and only if} \quad t \cdot b\xi = u,$$

and if this is the case, then let

$$(a, t) \circ (b, u) = (ab, u).$$

This definition is correct since $(ab)\xi \,\mathcal{L}\, b\xi \,\mathcal{L}\, u$. Furthermore, we define an action of T on $\mathcal{X}$ such that, for every $(a, t) \in \mathcal{X}$ and $v \in T$, let

$${}^{v}(a, t) = (a, vt).$$

This definition is also correct since $a\xi \,\mathcal{L}\, t \,\mathcal{L}\, vt$. Moreover, it is, indeed, an action since if $(a, t) \circ (b, u)$ is defined then ${}^{v}(a, t) \circ {}^{v}(b, u)$ is also defined and is equal to ${}^{v}((a, t) \circ (b, u))$.

For brevity, if $x, y \in \mathcal{X}$ and $x \circ y$ is defined then we will write that $\exists x \circ y$. More generally, if $f = f(v_1, \dots, v_n) \in F_n$ where F_n is the set of all groupoid terms in which the variables $v_1, \dots, v_n$ appear exactly once and in this order then we mean by writing $\exists f(x_1, \dots, x_n)$ for some $x_1, \dots, x_n \in \mathcal{X}$ that all products in $f(x_1, \dots, x_n)$ are defined. Denote the element $(\dots((v_1 \circ v_2) \circ v_3) \circ v_4 \dots) \circ v_n$ in F_n simply by $v_1 \circ v_2 \circ \dots \circ v_n$.

It is easy to check that the partial groupoid $(\mathcal{X}, \circ)$ is associative in the strictest sense possible: for any $x, y, z \in \mathcal{X}$, either of $\exists (x \circ y) \circ z$ or $\exists x \circ (y \circ z)$ is equivalent to requiring that both $\exists x \circ y$ and $\exists y \circ z$, and if this is the case then $(x \circ y) \circ z = x \circ (y \circ z)$. For, let $x = (a, t)$, $y = (b, u)$ and $z = (c, v)$. If $\exists (x \circ y) \circ z$ then $u \cdot c\xi = v$ since $x \circ y = (ab, u)$. Therefore $\exists y \circ z$. If $\exists x \circ (y \circ z)$ then we have $u \cdot c\xi = v$ and $t \cdot b\xi \cdot c\xi = t \cdot (bc)\xi = v$. Hence $t \cdot b\xi \cdot c\xi = u \cdot c\xi$ which implies $t \cdot b\xi = u$ since $b\xi \,\mathcal{L}\, u$. Thus $\exists x \circ y$. Conversely, it is clear that if $\exists x \circ y$ and $\exists y \circ z$ then we have $t \cdot b\xi = u$ and $u \cdot c\xi = v$. This implies that $\exists (x \circ y) \circ z$ and, since $t \cdot (bc)\xi = v$, also $\exists x \circ (y \circ z)$.

More generally, the standard proof of general associativity can be adapted to this situation, and one gets for $(\mathcal{X}, \circ)$ the following general form of associativity.

Lemma 4.2. *For any $x_1, x_2, \dots, x_n \in \mathcal{X}$ and $f \in F_n$, we have $\exists f(x_1, \dots, x_n)$ if and only if $\exists x_1 \circ x_2$, $\exists x_2 \circ x_3$, $\dots$, $\exists x_{n-1} \circ x_n$, and if this is the case then $f(x_1, \dots, x_n) = x_1 \circ x_2 \circ \dots \circ x_n$.*

Consider the free semigroup $\mathcal{X}^+$ on the set $\mathcal{X}$. For any $w = x_1x_2\dots x_n \in \mathcal{X}^+$ with $x_1, x_2, \dots, x_n \in \mathcal{X}$, we write $\prod w$ to denote $x_1 \circ x_2 \circ \cdots \circ x_n$. It is straightforward to check that if $\exists \prod w$ then

$$\ell\Bigl(\prod w\Bigr) = \prod_{i=1}^{n} \ell(x_i) \quad \text{and} \quad \omega\Bigl(\prod w\Bigr) = \omega(x_n).$$

Let $\mathbf{V}$ be any variety of normal bands containing $\mathbf{S}$. Define the relation $\tau_{\mathbf{V}}$ to be the congruence on $\mathcal{X}^+$ generated by the relation

$$\bigl\{(xy, z)\colon\ x, y, z \in \mathcal{X},\ \exists x \circ y \text{ and } z = x \circ y\bigr\} \cup \bigl\{\bigl(r^2, r\bigr)\colon\ r \in \mathcal{X}^+\bigr\} \cup \rho_{\mathbf{V}}, \tag{1}$$

where

$$\rho_{\mathbf{LN}} = \bigl\{(prsq, psrq)\colon\ p, r, s \in \mathcal{X}^+ \text{ and } q \in \mathcal{X}^*\bigr\},$$

$\rho_{\mathbf{RN}}$ is defined dually,

$$\rho_{\mathbf{N}} = \rho_{\mathbf{LN}} \cap \rho_{\mathbf{RN}} \quad \text{and} \quad \rho_{\mathbf{S}} = \rho_{\mathbf{LN}} \cup \rho_{\mathbf{RN}}.$$

Put $K_{\mathbf{V}} = \mathcal{X}^+/\tau_{\mathbf{V}}$. Obviously, we have $K_{\mathbf{V}} \in \mathbf{V}$. The action of T on $\mathcal{X}$ induces an action of T on $\mathcal{X}^+$. This action preserves $\tau_{\mathbf{V}}$ since it clearly preserves the relation generating $\tau_{\mathbf{V}}$. Thus an action of T on $K_{\mathbf{V}}$ is induced: for every $v \in T$ and $q \in \mathcal{X}^+$, we define ${}^{v}(q\tau_{\mathbf{V}}) = ({}^{v}q)\tau_{\mathbf{V}}$. This determines a restricted semidirect product $K_{\mathbf{V}} *_{rr} T$. Consider the mapping

$$\kappa_{\mathbf{V}} : S \to K_{\mathbf{V}} *_{rr} T, \qquad s\kappa_{\mathbf{V}} = \bigl((s, s\xi)\tau_{\mathbf{V}}, s\xi\bigr).$$

Lemma 4.3. *The mapping $\kappa_{\mathbf{V}}$ is a homomorphism.*

Proof. Let $s, t \in K_{\mathbf{V}} *_{rr} T$. Then we have

$$\begin{aligned}
s\kappa_{\mathbf{V}} \cdot t\kappa_{\mathbf{V}} &= \bigl((s, s\xi)\tau_{\mathbf{V}}, s\xi\bigr)\bigl((t, t\xi)\tau_{\mathbf{V}}, t\xi\bigr) \\
&= \bigl((s, s\xi)\tau_{\mathbf{V}} \cdot {}^{s\xi}\bigl((t, t\xi)\tau_{\mathbf{V}}\bigr), s\xi \cdot t\xi\bigr) \\
&= \bigl((s, s\xi)\tau_{\mathbf{V}} \cdot (t, s\xi \cdot t\xi)\tau_{\mathbf{V}}, s\xi \cdot t\xi\bigr) \\
&= \bigl((st, s\xi \cdot t\xi)\tau_{\mathbf{V}}, s\xi \cdot t\xi\bigr) \\
&= \bigl((st, (st)\xi)\tau_{\mathbf{V}}, (st)\xi\bigr) \\
&= (st)\kappa_{\mathbf{V}}. \quad \square
\end{aligned}$$

Now we associate a normal band variety to S in accordance with the statement of Theorem 4.1. Let $\mathbf{U} = \mathbf{S}$ if S is straight, let $\mathbf{U} = \mathbf{LN}$ [$\mathbf{U} = \mathbf{RN}$] if S is left [right] straight and let $\mathbf{U} = \mathbf{N}$ otherwise.

The sketch of the proof of Theorem 4.1 will be the following. The first and, in fact, the main step is to show that the kernel of the homomorphism $\kappa_{\mathbf{U}_0}$ is contained in $\mathcal{R} \cap \xi$. Then, by making use of $\kappa_{\mathbf{U}_0}$ and its dual, we give a homomorphism of S into a restricted semidirect product of a member of $\mathbf{U}$ by T whose kernel is contained in $\mathcal{H} \cap \xi$. Finally, we show that $\mathcal{H} \cap \xi$ is the equality relation which has the consequence that the latter homomorphism is an embedding.

Lemma 4.4. *The kernel of the homomorphism $\kappa_{\mathbf{U}_0}$ is contained in $\mathcal{R} \cap \xi$.*

Proof. Let $s, t \in S$ such that $s\kappa_{\mathbf{U}_0} = t\kappa_{\mathbf{U}_0}$. Then $s\xi = t\xi$ in T—denote it briefly by u—and $(s, u)\, \tau_{\mathbf{U}_0}\, (t, u)$ in $\mathcal{X}^+$. This means that there exists a sequence

$$(s, u) = w_0, w_1, \ldots, w_n = (t, u) \tag{2}$$

of words in $\mathcal{X}^+$ such that $w_{j+1},\ j = 0, 1, \ldots, n-1$, is obtained from w_j by one of the following steps:

(S1) $w_j = pzq$, $w_{j+1} = pxyq$ for some $p, q \in \mathcal{X}^*$ and $x, y, z \in \mathcal{X}$ where $\exists x \circ y$ and $z = x \circ y$;
(S1′) $w_j = pxyq$, $w_{j+1} = pzq$ for some $p, q \in \mathcal{X}^*$ and $x, y, z \in \mathcal{X}$ where $\exists x \circ y$ and $z = x \circ y$;
(S2) $w_j = prq$, $w_{j+1} = pr^2q$ for some $p, q \in \mathcal{X}^*$ and $r \in \mathcal{X}^+$;
(S2′) $w_j = pr^2q$, $w_{j+1} = prq$ for some $p, q \in \mathcal{X}^*$ and $r \in \mathcal{X}^+$;
$(\mathrm{S3})^{\mathbf{U}_0}$ $w_j\, \rho_{\mathbf{U}_0}\, w_{j+1}$.

Let us choose and fix an inverse s' of s and, for brevity, denote the idempotent $(ss')\xi$ in E_T by f.

Now we prove that the words $w_j,\ j = 0, 1, \ldots, n$, in (2) can be extended to words $\overrightarrow{w_j}$ in such a way that $\exists \prod \overrightarrow{w_j}$, and this product is constant for the sequence (2). This property will be very important in proving Lemma 4.4.

Lemma 4.5.

(i) *If $\mathbf{U}_0 = \mathbf{LN}$ then, for any $w_j = x_{j1}x_{j2}\ldots x_{jm_j},\ j = 0, 1, \ldots, n$ $(x_{jl} \in \mathcal{X},\ l = 1, 2, \ldots, m_j)$, there exists $\overrightarrow{w_j} = x_{j1}y_{j1}x_{j2}y_{j2}\ldots x_{jm_j}y_{jm_j} \in \mathcal{X}^+$ with $y_{j1}, y_{j2}, \ldots, y_{jm_j}$ being empty or from $\mathcal{X}$ such that $\exists \prod \overrightarrow{w_j}$, $\omega(\prod \overrightarrow{w_j}) = u$ and, for every $j,\ j = 0, 1, \ldots, n-1$, the equality $\ell(\prod \overrightarrow{w_j}) = \ell(\prod \overrightarrow{w_{j+1}})$ holds.*

(ii) *If $\mathbf{U}_0 = \mathbf{S}$ then, for any $w_j = x_{j1}x_{j2}\ldots x_{jm_j},\ j = 0, 1, \ldots n$ $(x_{jl} \in \mathcal{X},\ l = 1, 2, \ldots, m_j)$, there exists $\overrightarrow{w_j} = y_{j0}x_{j1}y_{j1}x_{j2}y_{j2}\ldots x_{jm_j}y_{jm_j} \in \mathcal{X}^+$ with $y_{j0}, y_{j1}, \ldots, y_{jm_j}$ being empty or from $\mathcal{X}$ such that $\exists \prod \overrightarrow{w_j}$,*

$$f \cdot \ell\big(h\big(\overrightarrow{w_j}\big)\big)\xi = \omega\big(h\big(\overrightarrow{w_j}\big)\big), \quad \omega\Big(\prod \overrightarrow{w_j}\Big) = u$$

and, for every $j,\ j = 0, 1, \ldots, n-1$, the equality $\ell(\prod \overrightarrow{w_j}) = \ell(\prod \overrightarrow{w_{j+1}})$ holds.

Proof. First we verify a technical lemma.

Lemma 4.6.

(i) *Let* $x=(a,t)$, $y=(b,u)$, $z=(c,v)\in\mathcal{X}$ *such that* $\exists x\circ y\circ z$.
  (a) *If* $b=b_1b_2$ *for some* $b_1,b_2\in S$ *then* $y_1=(b_1,t\cdot b_1\xi)$, $y_2=(b_2,u)\in\mathcal{X}$ *such that* $\exists x\circ y_1\circ y_2\circ z$ *and* $\ell(x\circ y\circ z)=\ell(x\circ y_1\circ y_2\circ z)$.
  (b) *Assume that* $b\in E_S$ *such that* $a\xi\,\mathcal{L}\,b\xi$. *Let* $b'\in E_S$ *such that* $b'\xi\,\mathcal{L}\,b\xi$ *and* $abc=ab'c$. *Then* $t=u$, $y'=(b',t)\in\mathcal{X}$, $\exists x\circ y'\circ z$ *and* $\ell(x\circ y\circ z)=\ell(x\circ y'\circ z)$.

(ii) *Let* $t\in T$ *and* $y=(b,u)$, $z=(c,v)\in\mathcal{X}$ *such that* $t\cdot b\xi=u$ *and* $\exists y\circ z$.
  (a) *If* $b=b_1b_2$ *for some* $b_1,b_2\in S$ *then* $y_1=(b_1,t\cdot b_1\xi)$, $y_2=(b_2,u)\in\mathcal{X}$ *such that* $\exists y_1\circ y_2\circ z$ *and* $\ell(y\circ z)=\ell(y_1\circ y_2\circ z)$.
  (b) *Assume that* $b\in E_S$ *such that* $t\,\mathcal{L}\,b\xi$. *Let* $b'\in E_S$ *such that* $b'\xi\,\mathcal{L}\,b\xi$ *and* $bc=b'c$. *Then* $t=u=t\cdot b'\xi$, $y'=(b',t)\in\mathcal{X}$, $\exists y'\circ z$ *and* $\ell(y\circ z)=\ell(y'\circ z)$.

Proof. Notice that in all statements, the equality of labels follows if we verify that the respective products exist.

(a) Clearly, $y_1\in\mathcal{X}$ and $\exists x\circ y_1$ in case (i). Since $t\cdot b\xi=u$ (in case (i) this follows from $\exists x\circ y$), we have $t\cdot b_1\xi\cdot b_2\xi=t\cdot(b_1b_2)\xi=t\cdot b\xi=u$. Therefore $y_2\in\mathcal{X}$ and $\exists y_1\circ y_2$. Finally, $\exists y_2\circ z$ because $\exists y\circ z$ and $\omega(y_2)=\omega(y)$.

(b) Now the equality $t\cdot b\xi=u$ and the assumptions on b and b' imply that $t\cdot b'\xi=t=t\cdot b\xi=u$. Therefore $y'\in\mathcal{X}$ and $\exists x\circ y'$ in case (i). Also $\exists y'\circ z$ since $\exists y\circ z$ and $\omega(y')=\omega(y)$. $\square$

If we verify the next lemma then Lemma 4.5 immediately follows by induction since $w_0=(s,u)$, and so $\overrightarrow{w_0}$ can also be chosen to be (s,u).

Lemma 4.7.

(i) *Suppose that* $\mathbf{U}_0=\mathbf{LN}$. *Let* $w,\overrightarrow{w}\in\mathcal{X}^+$ *such that*

$$w=x_1x_2\dots x_m \quad \textit{and} \quad \overrightarrow{w}=x_1y_1x_2y_2\dots x_my_m$$

where $m\geqslant 1$, $x_1,x_2,\dots,x_m\in\mathcal{X}$ *and each of* $y_1,y_2,\dots,y_m$ *is either empty or from* $\mathcal{X}$ *such that* $\exists\prod\overrightarrow{w}$ *and* $\omega(\prod\overrightarrow{w})=u$. *If* $w'=x_1'x_2'\dots x_{m'}'\in\mathcal{X}^+$ *is obtained from* w *by one of the steps* (S1)–(S3)$^{\mathbf{U}_0}$ *then there exists* $\overrightarrow{w'}=x_1'y_1'x_2'y_2'\dots x_{m'}'y_{m'}'\in\mathcal{X}^+$ *with* $y_1',y_2',\dots,y_{m'}'$ *being empty or from* $\mathcal{X}$ *such that* $\exists\prod\overrightarrow{w'}$,

$$\omega\Big(\prod\overrightarrow{w'}\Big)=u \quad \textit{and} \quad \ell\Big(\prod\overrightarrow{w}\Big)=\ell\Big(\prod\overrightarrow{w'}\Big).$$

(ii) *Assume that* $\mathbf{U}_0=\mathbf{S}$. *Let* $w,\overrightarrow{w}\in\mathcal{X}^+$ *such that*

$$w=x_1x_2\dots x_m \quad \textit{and} \quad \overrightarrow{w}=y_0x_1y_1x_2y_2\dots x_my_m$$

where $m \geqslant 1, x_1, x_2, \ldots, x_m \in \mathcal{X}$ and each of $y_0, y_1, \ldots, y_m$ is either empty or from $\mathcal{X}$ such that $\exists \prod \vec{w}$,

$$f \cdot \ell\big(h(\vec{w})\big)\xi = \omega\big(h(\vec{w})\big) \quad \textit{and} \quad \omega\Big(\prod \vec{w}\Big) = u.$$

If $w' = x'_1 x'_2 \ldots x'_{m'} \in \mathcal{X}^+$ is obtained from w by one of the steps (S1)–(S3)$^{\mathbf{U}_0}$ *then there exists $\overrightarrow{w'} = y'_0 x'_1 y'_1 x'_2 y'_2 \ldots x'_{m'} y'_{m'} \in \mathcal{X}^+$ with $y'_0, y'_1, \ldots, y'_{m'}$ being empty or from $\mathcal{X}$ such that $\exists \prod \overrightarrow{w'}$,*

$$f \cdot \ell\big(h(\overrightarrow{w'})\big)\xi = \omega\big(h(\overrightarrow{w'})\big), \quad \omega\Big(\prod \overrightarrow{w'}\Big) = u \quad \textit{and} \quad \ell\Big(\prod \vec{w}\Big) = \ell\Big(\prod \overrightarrow{w'}\Big).$$

Proof. If p is the segment $x_k x_{k+1} \ldots x_{k+l}$, $k \geqslant 0$, $0 \leqslant l \leqslant m-k$, of w then $\tilde{p}$ will denote the segment $x_k y_k x_{k+1} y_{k+1} \ldots y_{k+l-1} x_{k+l}$ of $\vec{w}$. In particular, if p is the empty initial segment of w then $\tilde{p}$ is empty, and if we write $\vec{w}$ as $\overline{p\vec{w}} = \tilde{p} y \vec{w}$ then y is also supposed to be empty.

(i) We distinguish five cases according to the rules applied to obtain w' from w. Roughly speaking, the proof goes in each (more complicated) case in the following way: starting with $\prod \vec{w}$, we apply the steps given in Lemma 4.6(i) combined with forming products (see Lemma 4.2) to obtain $\prod \overrightarrow{w'}$. Due to Lemma 4.6(i), this procedure can be completely controlled by making calculations with the labels. Therefore only the latter will be given in details, and it will be left to the reader to check by Lemmas 4.2 and 4.6(i) that all elements and products induced exist in $\mathcal{X}$.

(S1) Let $w = pzq$ and $w' = pxyq$ where $p, q \in \mathcal{X}^*$ and $x, y, z \in \mathcal{X}$ with $z = x \circ y$. If $\vec{w} = \tilde{p} y_1 z y_2 \tilde{q} y_3$ then we simply define $\overrightarrow{w'} = \tilde{p} y_1 x y y_2 \tilde{q} y_3$.

(S1′) Now $w = pxyq$ and $w' = pzq$ where $p, q \in \mathcal{X}^*$ and $x, y, z \in \mathcal{X}$ with $z = x \circ y$. Let $\vec{w} = \tilde{p} y_1 x y_2 y y_3 \tilde{q} y_4$. Put $x = (a, t)$, $y = (b, v)$ and $y_j = (d_j, s_j)$, $j = 1, 2, 3, 4$. By assumptions, we have $\exists x \circ y$, $\exists x \circ y_2$ and $\exists y_2 \circ y$. Therefore $t \cdot b\xi = v = s_2 \cdot b\xi = t \cdot d_2\xi \cdot b\xi$. Since $t \,\mathcal{L}\, a\xi$, hence we obtain that $a\xi \cdot b\xi = a\xi \cdot d_2\xi \cdot b\xi$, that is, $ab \,\xi\, ad_2b$. This implies that, for any $g \in V(ad_2b)$, the element bga is idempotent, since $g\xi \in V((ad_2b)\xi) = V((ab)\xi)$ and ξ is idempotent pure. Clearly, d_2bga is also idempotent and $(d_2bga)\xi \,\mathcal{L}\, (a'a)\xi \,\mathcal{L}\, (bga)\xi$ for any $a' \in V(a)$. Therefore the idempotents $a'a, d_2bga$ and bga belong to a normal band E_λ, $\lambda \in \Lambda$. Hence it follows that

$$\begin{aligned}
\ell\Big(\prod \vec{w}\Big) &= \ell\Big(\prod \tilde{p}\Big) \cdot d_1 a d_2 b d_3 \cdot \ell\Big(\prod \tilde{q}\Big) \cdot d_4 \\
&= \ell\Big(\prod \tilde{p}\Big) \cdot d_1 \cdot ad_2b \cdot g \cdot ad_2b \cdot d_3 \cdot \ell\Big(\prod \tilde{q}\Big) \cdot d_4 \\
&= \ell\Big(\prod \tilde{p}\Big) \cdot d_1 a \cdot a'a \cdot d_2bga \cdot bga \cdot a'a \cdot d_2bd_3 \cdot \ell\Big(\prod \tilde{q}\Big) \cdot d_4 \\
&= \ell\Big(\prod \tilde{p}\Big) \cdot d_1 a \cdot a'a \cdot bga \cdot d_2bga \cdot a'a \cdot d_2bd_3 \cdot \ell\Big(\prod \tilde{q}\Big) \cdot d_4 \\
&= \ell\Big(\prod \tilde{p}\Big) \cdot d_1 ab \cdot gad_2bd_3 \cdot \ell\Big(\prod \tilde{q}\Big) \cdot d_4.
\end{aligned}$$

Therefore we can define $\overrightarrow{w'} = \tilde{p} y_1 z z_2 \tilde{q} y_4$ where $z_2 = (gad_2bd_3, v \cdot d_3\xi)$.

(S2) In this case, we have $w = prq$ and $w' = pr^2q$ where $p, q \in \mathcal{X}^*$ and $r \in \mathcal{X}^+$. If $\vec{w} = \tilde{p}y_1\tilde{r}y_2\tilde{q}y_3$ then define $\overrightarrow{w'} = \tilde{p}y_1\tilde{r}y_4\tilde{r}y_2\tilde{q}y_3$ where $y_4 = (g, \omega(\prod\tilde{r}) \cdot g\xi)$ for some $g \in V(\ell(\prod\tilde{r}))$.

(S2′) Now $w = pr^2q$ and $w' = prq$ for some $p, q \in \mathcal{X}^*$ and $r \in \mathcal{X}^+$. If $\vec{w} = \tilde{p}y_1\tilde{r}y_2\tilde{\tilde{r}}y_3\tilde{q}y_4$ (here $\tilde{\tilde{r}}$ indicates that the segments of $\vec{w}$ corresponding to the two segments r in w might be different) then define $\overrightarrow{w'}$ simply to be $\tilde{p}y_1\tilde{r}z_2\tilde{q}y_4$ where $z_2 = y_2 \circ \prod\tilde{\tilde{r}} \circ y_3$.

(S3)$^{\mathbf{U}_0}$ Suppose that $w = prsq$ and $w' = psrq$ where $p, s, r \in \mathcal{X}^+$ and $q \in \mathcal{X}^*$. Let $\vec{w} = \tilde{p}y_1\tilde{r}y_2\tilde{s}y_3\tilde{q}y_4$. Put $\prod\tilde{r} = (a, t)$, $\prod\tilde{s} = (b, v)$ and $y_j = (d_j, s_j)$, $j = 1, 2, 3, 4$. Choose any $g \in V(ad_2b)$. Then we have

$$\begin{aligned}\ell\Big(\prod\vec{w}\Big) &= \ell\Big(\prod\tilde{p}\Big) \cdot d_1ad_2bd_3 \cdot \ell\Big(\prod\tilde{q}\Big) \cdot d_4\\ &= \ell\Big(\prod\tilde{p}\Big) \cdot d_1ad_2 \cdot bga \cdot d_2bd_3 \cdot \ell\Big(\prod\tilde{q}\Big) \cdot d_4.\end{aligned}$$

Define $\overrightarrow{w'} = \tilde{p}z_1\tilde{s}z_2\tilde{r}z_3\tilde{q}y_4$ where $z_1 = (d_1ad_2, s_2)$, $z_2 = (g, v \cdot g\xi)$ and $z_3 = (d_2bd_3, s_3)$.

(ii) The proof is very similar to the proof of (i). If necessary, we apply Lemma 4.6(ii) instead of Lemma 4.6(i). This completes the proof of Lemma 4.7 and also of Lemma 4.5. □

It is the time to turn to proving Lemma 4.4. Since $s\xi = t\xi$, all we have to prove is that $s\,\mathcal{R}\,t$. Lemma 4.5 ensures that $s = \ell((s, u)) = \ell(\prod\overrightarrow{w_0}) = \ell(\prod\overrightarrow{w_n})$ where

$$\overrightarrow{w_n} = \begin{cases}(t, u)y_1 & \text{if } \mathbf{U}_0 = \mathbf{LN},\\ y_0(t, u)y_1 & \text{if } \mathbf{U}_0 = \mathbf{S}.\end{cases}$$

If $\mathbf{U}_0 = \mathbf{LN}$, or $\mathbf{U}_0 = \mathbf{S}$ and y_0 is empty then $\overrightarrow{w_n} = (t, u)y_1$, and we clearly have $s = t \cdot \ell(y_1) \leqslant_{\mathcal{R}} t$. Now let $\mathbf{U}_0 = \mathbf{S}$ and $\overrightarrow{w_n} = y_0(t, u)y_1$ where y_0 is non-empty. Then $\mathbf{U} \subseteq \mathbf{RN}$, and so E_i is a right normal band for any $i \in I$. Moreover, we see that $s = a_0ta_1 = ss'a_0ta_1$ where $a_j = \ell(y_j)$, $j = 0, 1$. By assumption, $(ss')\xi \cdot a_0\xi = f \cdot a_0\xi = \omega(y_0)$ and $\omega(y_0) \cdot t\xi = u$. Thus $(ss')\xi \cdot a_0\xi \cdot t\xi = u = t\xi$. For any $g \in V(ss'a_0t)$, we see that $g\xi \in V((ss'a_0t)\xi) = V(t\xi)$. Therefore $ss'a_0tg$ and tg are idempotents in S and $(ss'a_0t)\xi\,\mathcal{R}\,(ss')\xi = f\,\mathcal{R}\,u\,\mathcal{R}\,(tg)\xi$. This implies that $ss'a_0tg, tg \in E_i$ for some $i \in I$. Since E_i is a right normal band, we obtain that

$$\begin{aligned}s &= ss'a_0ta_1 = ss'a_0t \cdot g \cdot ss'a_0ta_1 = ss'a_0tg \cdot tg \cdot ss'a_0ta_1\\ &= tg \cdot ss'a_0tg \cdot tg \cdot ss'a_0ta_1 \leqslant_{\mathcal{R}} t.\end{aligned}$$

Since the roles of s and t can be changed, the relation $s\,\mathcal{R}\,t$ yields. This completes the proof of Lemma 4.4. □

Let $T = \mathcal{M}[G; I, \Lambda; P]$ where P is a normal sandwich matrix. Applying Theorem 3.1 and Lemma 4.4, we obtain a homomorphism ϕ of S into a Pastijn product $B \odot T$ where $B \in \mathbf{U}_0$, G acts on B, $\phi\pi_2^{B\odot T} = \xi^\natural$ and $\ker\phi \subseteq \mathcal{R} \cap \xi$. By a dual argument we can find a homomorphism ψ of S into a dual Pastijn product $T \odot_d C$ where $C \in \mathbf{U}_1$, G acts on C by automorphisms on the right, $\psi\pi_1^{T\odot_2 C} = \xi^\natural$ and $\ker\psi \subseteq \mathcal{L} \cap \xi$. Now Theorem 3.3 ensures that there exists a homomorphism ψ' of S into a Pastijn product $C \odot T$ where G acts on C, $\psi'\pi_2^{C\odot T} = \xi^\natural$ and $\ker\psi' \subseteq \mathcal{L} \cap \xi$.

Consider the direct product $B \times C$. Clearly, $B \times C \in \mathbf{U}$ and the actions of G on B and on C naturally induce an action of G on $B \times C$ by putting ${}^g(b,c) = ({}^g b, {}^g c)$ for every $g \in G$, $b \in B$ and $c \in C$. This action defines a Pastijn product $(B \times C) \odot T$, and the homomorphisms ϕ and ψ' determine a homomorphism $\Phi : S \to (B \times C) \odot T$ by the rule $s\Phi = ((s\phi\pi_1^{B\odot T}, s\psi'\pi_1^{C\odot T}), s\xi)$. Clearly, we have $\ker\Phi = \ker\phi \cap \ker\psi' \subseteq (\mathcal{R} \cap \xi) \cap (\mathcal{L} \cap \xi) = \mathcal{H} \cap \xi$.

Lemma 4.8. *The relation $\mathcal{H} \cap \xi$ is the equality relation on S.*

Proof. Let $a, b \in S$ with $a\,\mathcal{H} \cap \xi\, b$. Choose $a' \in V(a)$ and $b' \in V(b)$. The relation $a\,\mathcal{R}\,b$ implies that $a = aa' \cdot bb' \cdot a$. Since $a\,\xi\,b$, we obtain that $(a'b)\xi$ and $(ba')\xi$ are idempotents in T. The congruence ξ is idempotent pure, therefore $a'b, ba' \in E_S$. Moreover, we have $(aa')\xi = (ba')\xi\,\mathcal{R}\,(bb')\xi$ in E_T, and so $aa', ba', bb' \in E_i$ for some $i \in I$. Since E_i is a normal band, we infer that

$$a = a \cdot a'b \cdot a'b \cdot b'a = aa' \cdot ba' \cdot bb' \cdot aa' \cdot a$$
$$= aa' \cdot bb' \cdot ba' \cdot aa' \cdot a = aa'ba'a = ba'a.$$

However, also $a\,\mathcal{L}\,b$ which implies $a = ba'a = b$, completing the proof. □

This lemma ensures that $\ker\Phi$ is the equality relation on S whence it follows that Φ is injective. Thus Φ is an embedding of S in a Pastijn product of a band in $\mathbf{U}$ by T. Theorem 3.2 implies that S is embeddable in a restricted semidirect product of a band in $\mathbf{U}$ by T. This completes the proof of Theorem 4.1. □

Now we can easily establish the assertion stated in the introduction on weakly E-unitary locally inverse semigroups with injective structure mappings.

Proposition 4.9. *The following conditions are equivalent for any weakly E-unitary locally inverse semigroup S:*

(i) *S has injective structure mappings,*
(ii) *both relations $\mathcal{R} \cap \xi$ and $\mathcal{L} \cap \xi$ are equal to the equality relation on S,*
(iii) *S is straight.*

Proof. In [7], statements (i) and (ii) are proved to be equivalent for any locally inverse semigroup S.

(ii) ⇒ (iii) Let $\widehat{E}$ be an idempotent ξ-class of S. Since ξ is idempotent pure, $\widehat{E}$ is a normal subband in E. If $\mathcal{R} \cap \xi$ is the equality relation on S then $\mathcal{R}$ is the equality relation on $\widehat{E}$, and so $\widehat{E}$ is left normal. Dually, it is also right normal, whence we see that $\widehat{E}$ is a semilattice. Thus S is straight.

(iii) ⇒ (ii) We have seen in the proof of Lemma 4.8 that $a = ba'a$ for any $a, b \in S$ with $a\,\mathcal{R} \cap \xi\, b$ and for any $a' \in V(a)$. Since S is straight and $a'a, a'b$ are ξ-related idempotents in S, they commute. Therefore we have $a = ba'a = aa'ba'a = aa'aa'b = aa'b = b$. This shows that $\mathcal{R} \cap \xi$ is the equality relation. Dually, it follows that $\mathcal{L} \cap \xi$ is also the equality relation. $\square$

For completeness we prove that $\kappa_{\mathbf{U}}$ is an embedding. In fact, this will follow from a universal property of $\kappa_{\mathbf{U}}$.

The following homomorphisms play a crucial role here. Let $L, \overline{L}$ be any bands and let $U, \overline{U}$ be completely simple semigroups acting on L and $\overline{L}$, respectively. If $\phi_1 : L \to \overline{L}$ and $\phi_2 : U \to \overline{U}$ are homomorphisms such that, for every $l \in L$ and $u \in U$, we have $({}^u l)\phi_1 = {}^{u\phi_2}(l\phi_1)$, then the mapping $\phi : L *_{rr} U \to \overline{L} *_{rr} \overline{U}$ defined by $(l, u)\phi = (l\phi_1, u\phi_2)$ is easily seen to be a homomorphism. If a homomorphism $\phi : L *_{rr} U \to \overline{L} *_{rr} \overline{U}$ is of this form for some ϕ_1 and ϕ_2 then we call it a *splitting homomorphism* of the restricted semidirect products, and denote it also by (ϕ_1, ϕ_2).

Proposition 4.10. *Let S be a weakly E-unitary locally inverse semigroup and let $\mathbf{V}$ be a variety of normal bands. Suppose that $L \in \mathbf{V}$, U is a completely simple semigroup acting on L and $\psi : S \to L *_{rr} U$ is a homomorphism. Then there exists a unique splitting homomorphism $\phi : K_{\mathbf{V}} *_{rr} T \to L *_{rr} U$ such that $\psi = \kappa_{\mathbf{V}}\phi$.*

Proof. For brevity, we denote the ith $(i = 1, 2)$ projections $\pi_i^{K_{\mathbf{V}} *_{rr} T}$ and $\pi_i^{L *_{rr} U}$ by π_i and $\bar{\pi}_i$, respectively. First we prove the uniqueness of ϕ. Assume that $\phi = (\phi_1, \phi_2) : K_{\mathbf{V}} *_{rr} T \to L *_{rr} U$ is a splitting homomorphism satisfying $\psi = \kappa_{\mathbf{V}}\phi$. Obviously, this equality is equivalent to the equalities $\psi\bar{\pi}_i = \kappa_{\mathbf{V}}\pi_i\phi_i$ $(i = 1, 2)$. It follows from the definition of $\kappa_{\mathbf{V}}$ that $\kappa_{\mathbf{V}}\pi_2 = \xi^{\natural}$. Thus ϕ_2 must satisfy the equality

$$\xi^{\natural}\phi_2 = \psi\bar{\pi}_2. \tag{3}$$

Hence we see that ϕ_2 is uniquely determined.

Moreover, the equality $\psi\bar{\pi}_1 = \kappa_{\mathbf{V}}\pi_1\phi_1$ is equivalent to the fact that

$$\big((a, a\xi)\tau_{\mathbf{V}}\big)\phi_1 = a\psi\bar{\pi}_1 \tag{4}$$

for every $a \in S$. Hence it follows that

$$\big((a, t)\tau_{\mathbf{V}}\big)\phi_1 = {}^{s\phi_2}\big(a\psi\bar{\pi}_1\big)$$

for any $(a, t) \in \mathcal{X}$ and for any $s \in T$ with $s \cdot a\xi = t$. For, if $(a, t) \in \mathcal{X}$ then $t\,\mathcal{L}\,a\xi$. Let $s \in T$ such that $s \cdot a\xi = t$. Then we have $(a, t) = {}^s(a, a\xi)$. Applying (4) and the fact that (ϕ_1, ϕ_2) is a splitting homomorphism, we see that

$$\begin{aligned}\big((a, t)\tau_{\mathbf{V}}\big)\phi_1 &= \big(\big({}^s(a, a\xi)\big)\tau_{\mathbf{V}}\big)\phi_1 = \big({}^s\big((a, a\xi)\tau_{\mathbf{V}}\big)\big)\phi_1 \\ &= {}^{s\phi_2}\big((a, a\xi)\tau_{\mathbf{V}}\big)\phi_1 = {}^{s\phi_2}\big(a\psi\bar{\pi}_1\big).\end{aligned}$$

Since the set $\{(a, t)\tau_{\mathbf{V}}\colon (a, t) \in \mathcal{X}\}$ generates the band $K_{\mathbf{V}}$, we obtain that ϕ_1 is uniquely determined.

The existence of ϕ will follow if we show that the only pair (ϕ_1, ϕ_2) possible by the above argument exists and forms a splitting homomorphism. Since $\psi\bar{\pi}_2$ is a homomorphism of S into a completely simple semigroup and ξ is the least completely simple congruence on S, we immediately see that the homomorphism ϕ_2 of T into U satisfying (3) exists.

Now let $\phi_2 : T \to U$ be the unique homomorphism such that (3) is valid. Observe that if $a \in S$ and $s_1, s_2, t \in T$ such that $t = s_1 \cdot a\xi = s_2 \cdot a\xi$ then ${}^{s_1\phi_2}(a\psi\bar{\pi}_1) = {}^{s_2\phi_2}(a\psi\bar{\pi}_1)$. For, $a\psi \in L *_{rr} U$, and so ${}^{uu'}(a\psi\pi_1) = a\psi\pi_1$ for $u = a\psi\pi_2$ and for any $u' \in V(u)$. Since (3) and the assumption on s_1 and s_2 imply that $s_1\phi_2 \cdot uu' = s_1\phi_2 \cdot (a\xi)\phi_2 \cdot u' = t\phi_2 \cdot u' = s_2\phi_2 \cdot (a\xi)\phi_2 \cdot u' = s_2\phi_2 \cdot uu'$, the required equality follows. This allows us to define a mapping $\mathcal{X} \to L$ by putting $(a, t) \mapsto {}^{s\phi_2}(a\psi\bar{\pi}_1)$ where s is any element in T with $t = s \cdot a\xi$. Let θ be the unique extension of this mapping to a homomorphism of $\mathcal{X}^+$ to L. We intend to show that $\tau_{\mathbf{V}} \subseteq \ker\theta$. For this purpose, it suffices to verify that the relation in (1) is contained in $\ker\theta$. Let $(a, s), (b, t) \in \mathcal{X}$ such that $\exists(a, s) \circ (b, t)$. Then $s \cdot b\xi = t$ and $(a, s) \circ (b, t) = (ab, t)$. Let $s = x \cdot a\xi$ whence $t = x \cdot a\xi \cdot b\xi = x \cdot (ab)\xi$. Thus, we have

$$\begin{aligned}\big((a,s)(b,t)\big)\theta &= (a,s)\theta \cdot (b,t)\theta = {}^{x\phi_2}\big(a\psi\bar{\pi}_1\big) \cdot {}^{(x\cdot a\xi)\phi_2}\big(b\psi\bar{\pi}_1\big)\\ &= {}^{x\phi_2}\big(a\psi\bar{\pi}_1 \cdot {}^{a\psi\bar{\pi}_2}\big(b\psi\bar{\pi}_1\big)\big) = {}^{x\phi_2}\big((a\psi \cdot b\psi)\bar{\pi}_1\big) = {}^{x\phi_2}\big((ab)\psi\bar{\pi}_1\big)\\ &= (ab,t)\theta.\end{aligned}$$

This implies that the first set in (1) is contained in $\ker\theta$. The same inclusion for the other two sets follows from the assumption that $L \in \mathbf{V}$. Thus the existence of a homomorphism ϕ_1 with $\tau_{\mathbf{V}}^{\natural}\phi_1 = \theta$ follows, and so $\phi = (\phi_1, \phi_2)$ satisfies the condition that $\psi = \kappa_{\mathbf{V}}\phi$.

To complete the proof, it remains to show that

$$\big({}^{t}k\big)\phi_1 = {}^{t\phi_2}(k\phi_1) \quad (t \in T,\ k \in K_{\mathbf{V}}).$$

Taking into account the definition of the action of T on $K_{\mathbf{V}}$ and the definition of ϕ_1, it suffices to check that $({}^{t}(a, s))\theta = {}^{t\phi_2}((a, s)\theta)$. This equality easily follows: if $s = x \cdot a\xi$ then $ts = tx \cdot a\xi$, and so we have

$$\big({}^{t}(a,s)\big)\theta = (a,ts)\theta = {}^{(tx)\phi_2}\big(a\psi\bar{\pi}_1\big) = {}^{t\phi_2}\big({}^{x\phi_2}\big(a\psi\bar{\pi}_1\big)\big) = {}^{t\phi_2}\big((a,s)\theta\big).$$

This completes the proof. □

The universal property of $\kappa_{\mathbf{V}}$ formulated in Proposition 4.10 allows us to call $\kappa_{\mathbf{V}}$ the *canonical homomorphism of S into a restricted semidirect product of a band in* **V** *by a completely simple semigroup*.

Now let S be a weakly E-unitary locally inverse semigroup, and define the variety **U** of normal bands as in the proof of Theorem 4.1. It was proved above that there exists an embedding of S into a restricted semidirect product of a band in **U** by a completely simple semigroup. If ψ is such an embedding then, by applying Proposition 4.10 for this embedding, we obtain that $\psi = \kappa_{\mathbf{U}}\phi$ for some splitting homomorphism ϕ. This implies that $\kappa_{\mathbf{U}}$ is an embedding.

Corollary 4.11. *Let S be a weakly E-unitary locally inverse semigroup. Let $\mathbf{U} = \mathbf{S}$ if S is straight, let $\mathbf{U} = \mathbf{LN}$ $[\mathbf{U} = \mathbf{RN}]$ if S is left $[$right$]$ straight and let $\mathbf{U} = \mathbf{N}$ otherwise. Then $\kappa_{\mathbf{U}}$ is an embedding.*

References

[1] K. Auinger, L. Polák, A semidirect product for locally inverse semigroups, Acta Sci. Math. 63 (1997) 405–435.
[2] B. Billhardt, M.B. Szendrei, Associativity of regular semidirect product of existence varieties, J. Austral. Math. Soc. Ser. A 69 (2000) 85–115.
[3] J.M. Howie, Fundamentals of Semigroup Theory, in: London Math. Soc. Monogr. (N.S.), Vol. 12, Oxford Univ. Press, Oxford, 1995.
[4] P.R. Jones, P.G. Trotter, Semidirect products of regular semigroups, Trans. Amer. Math. Soc. 349 (1997) 4265–4310.
[5] J. Kad’ourek, On some existence varieties of locally inverse semigroups, Internat. J. Algebra Comput. 6 (1996) 761–788.
[6] J. Kad’ourek, On some existence varieties of locally orthodox semigroups, Internat. J. Algebra Comput. 7 (1997) 93–131.
[7] K.S.S. Nambooripad, R. Veeramony, Subdirect products of regular semigroups, Semigroup Forum 27 (1983) 265–307.
[8] F. Pastijn, Rectangular bands of inverse semigroups, Simon Stevin 56 (1982) 1–97.
[9] F. Pastijn, The structure of pseudo-inverse semigroups, Trans. Amer. Math. Soc. 273 (1982) 631–655.
[10] F. Pastijn, M. Petrich, Straight locally inverse semigroups, Proc. London Math. Soc. (3) 49 (1984) 307–328.
[11] M.B. Szendrei, E-unitary regular semigroups, Proc. Roy. Soc. Edinburgh A 106 (1987) 89–102.
[12] R. Veeramony, Proper pseudo-inverse semigroups, Simon Stevin 58 (1984) 65–86.